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Characterization of Factor Graph by Mooij’s Sufficient Condition for Convergence of the Sum-Product Algorithm

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SUMMARY Recently, Mooij et al. proposed new sufficient conditions for convergence of the sum-product algorithm, and it was also shown that if the factor graph is a tree, Mooij’s sufficient condition for convergence is always activated. In this letter, we show that the converse of the above statement is also true under some assumption, and that the assumption holds for the sum-product algorithm. These newly obtained fact implies that Mooij’s sufficient condition for convergence of the sum-product decoding is activated if and only if the factor graph of the a posteriori probability of the transmitted codeword is a tree.

key words: convergence of the sum-product algorithm, factor graph, message-passing, sum-product decoding, LDPC codes

1. Introduction

The sum-product algorithm is widely accepted as an efficient iterative algorithm for approximate inference on graphical models, and applications of the algorithm can be found in various areas. In the practical use of the sum-product algorithm, we sometimes encounter the following two undesired problems which may be mutually related: (1) there is no guarantee of convergence of the algorithm, (2) even if the algorithm converges, the accuracy of the result toward the exact marginals is unclear. The theoretical analyses of these two problems and their relation are not straightforward and plenty of studies have been devoted to these topics [2], [4], [7]–[9].

Recently, Mooij et al. [6] introduced new sufficient conditions for convergence of the sum-product algorithm. They were expressed in terms of the upper bound for the spectral radius of a particular matrix whose entries are defined by a probability distribution under consideration. Moreover, it was shown that if the factor graph corresponding to the probability distribution is a tree (recall that this is a well known sufficient condition for convergence of the sum-product algorithm), Mooij’s sufficient condition for convergence is always activated.

In this letter, we show that the converse of the above statement is also true under a certain assumption on the matrix mentioned above, which holds in one of very important applications. More precisely, under the assumption that the matrix is result in a zero-one matrix, we can show that if Mooij’s sufficient condition for convergence of the sum-product algorithm is activated, the factor graph must be a tree. Moreover, we show that the assumption always holds when the matrix are defined by a posterior probability of a transmitted codeword. This implies that Mooij’s sufficient condition for the sum-product decoding is activated if and only if the factor graph is a tree.

This letter is organized as follows. In the next section, we begin with a review of the sum-product algorithm and brief introduction of Mooij’s results. Section 3 is devoted to show the main theorem. Application of the main theorem to the sum-product decoding will be discussed in Sect. 4. Concluding remarks will be given in Sect. 5.

2. Mooij’s Sufficient Condition

2.1 Sum-Product Algorithm

Consider n random variables \( x_i \) \( (i = 1, 2, \ldots, n) \) on a discrete set \( X_i \). We are interested in the class of probability distributions \( p(x) \) on \( X := \prod_{i=1}^{n} X_i \) that can be expressed as

\[
p(x) := \frac{1}{Z} \prod_{\mu} \psi_{\mu}(x_{\mu})
\]  

where \( \mu \) denotes an index specifying a factor of \( p(x) \), \( x_{\mu} := (x_i)_{i \in \mu} \in \prod_{i \in \mu} X_i \) for a set of indices \( I_{\mu} \subset \{1, 2, \ldots, n\} \) and \( Z \) is a normalization constant.

Let \( \mathcal{F} \) and \( \mathcal{V} \) be disjoint sets of vertices and assume that elements of \( \mathcal{F} \) and \( \mathcal{V} \) are labeled with indices of factors \( \psi_{\mu} \) and random variables \( x_i \) of \( p(x) \) given in Eq. (1), respectively. Then the factor graph associated with \( p(x) \) is a bipartite graph with a vertex set \( \mathcal{F} \cup \mathcal{V} \) and an edge set \( E \subseteq \mathcal{F} \times \mathcal{V} \), where \( (\mu, i) \in E \) if and only if \( i \in I_{\mu} \). Vertices of \( \mathcal{F} \) and \( \mathcal{V} \) are called factor nodes and variable nodes, respectively.

For \( p(x) \), the marginal distribution \( p_i(x_i) \) of \( x_i \) can be efficiently calculated or approximated by the sum-product algorithm [1], which is executed by message-passing on the factor graph. For \( (\mu, i) \in E \) and \( x_i \in X_i \), let \( m_{\mu \rightarrow i}(x_i) \) and \( m_{i \rightarrow \mu}(x_i) \) denote messages passed from a factor node to a variable node and the other way round, respectively. Then the update rules for messages are given as

\[
\begin{align*}
\tilde{m}_{\mu \rightarrow i}(x_i) & \propto \sum_{\{x_j\} \in \mu \setminus \{i\}} \psi_{\mu}(x_{\mu}) \prod_{j \in \mu \setminus \{i\}} m_{j \rightarrow \mu}(x_j), \\
\tilde{m}_{i \rightarrow \mu}(x_i) & \propto \prod_{v \in N_i \setminus \mu} m_{v \rightarrow i}(x_i), 
\end{align*}
\]

\( \tilde{m}_{\mu \rightarrow i}(x_i) \) and \( \tilde{m}_{i \rightarrow \mu}(x_i) \) are the approximate messages that are used in the update rules for messages.
where \( N_i \) denotes a set of indices of factor nodes neighboring a variable node \( x_i \). In this letter, we consider the following update equation of messages that is obtained by combining above two update rules\(^1\):

\[
\bar{m}_{\mu \rightarrow \nu}(x_i) = k_{\mu \rightarrow \nu} \sum_{x_j \in \mathcal{N}_\nu \setminus \{\mu\}} \psi^\mu(x_j) \prod_{\mu \in \mathcal{N}_j \cup \{\nu\}} m_{\nu \rightarrow \mu}(x_j),
\]

(2)

where \( k_{\mu \rightarrow \nu} \) is a normalization constant. If all messages have converged to some fixed point \( m^* \), the approximate marginals are obtained by

\[
b_i(x_i) = k_i \prod_{\mu \in \mathcal{N}_i} m_{\mu \rightarrow i}^*(x_i)
\]

for \( i = 1, 2, \ldots, n \) and \( x_i \in \mathcal{X}_i \) where \( k_i \) is a normalization constant.

2.2 Sufficient Conditions for Convergence of the Sum-Product Algorithm

Let \( W \) be a finite-dimensional real vector space and \( \| \cdot \| \) be a norm on \( W \). Recall that a norm on \( W \) induces a metric \( d \) on \( W \) that is defined by \( d(u, u') := \| u - u' \| \) for \( u, u' \in W \), and the resulting metric space is complete. Let \((X, d)\) be a metric space. Then a mapping \( f : X \rightarrow X \) is called a contraction with respect to \( d \) if there exists \( 0 < K < 1 \) such that

\[
d(f(x), f(y)) \leq Kd(x, y), \quad \text{for all } x, y \in X.
\]

The following proposition is well known as the contraction principle.

**Proposition 1:** Let \( f : X \rightarrow X \) be a contraction of a complete metric space \((X, d)\). Then \( f \) has a unique fixed point \( x_\infty \in X \). Moreover, for any \( x_1 \in X \), the sequence \( x_1, x_2, \ldots \) given by \( x_{n+1} = f(x_n) \) converges to \( x_\infty \).

As we will present in Theorem 1 below, Mooij et al. [6] proposed sufficient conditions under which the update equation of messages given in Eq. (2) results in a contraction mapping with respect to a certain metric.

In the following discussion, let \( x_{\mu,i} := (x_i)_{i \in \mathcal{N}_\mu \setminus \{\mu\}} \) and \( x_{\mu,i,j} := (x_i)_{i \in \mathcal{N}_\mu \setminus \{i, j\}} \) for \( i, j \in \mathcal{I}_\mu \), and define \( h^\beta := h^\beta(x_j, x_{\mu,i,j}) \) as a function of \( x_j \) and \( x_{\mu,i,j} \). Moreover, for \( \psi^\mu \) in Eq. (1) and \( h^\beta \) defined above, denote \( \psi_{\alpha\beta\gamma}^\mu := \psi^\mu(x_i = \alpha, x_j = \beta, x_{\mu,i,j} = \gamma) \) and \( h_{\alpha\beta\gamma}^\mu := h^\beta(x_j = \beta, x_{\mu,i,j} = \gamma) \) for \( i, j \in \mathcal{I}_\mu \) to lighten the notation. For a square matrix \( A \), we denote by \( \rho(A) \) its spectral radius. For a set \( X \), we define the indicator function \( 1_X \) of \( X \) by

\[
1_X(x) :=
\begin{cases}
1, & \text{if } x \in X, \\
0, & \text{if } x \notin X.
\end{cases}
\]

**Theorem 1:** [6, Theorem 4] Assume that each factor \( \psi^\mu \) of \( p(x) \) given in Eq. (1) is a positive function. Let

\[
N(\psi^\mu, i, j) := \sup_{\beta, \gamma} \sup_{x_{\mu,i,j}} \frac{1}{4} \log \left( \frac{\psi_{\alpha\beta\gamma}^\mu}{\psi_{\alpha'\beta'\gamma'}^\mu} \right)
\]

(3)

Then, if \( \rho(A) < 1 \) for the matrix \( A = [A_{\mu \rightarrow i, \nu \rightarrow j}] \) defined as

\[
A_{\mu \rightarrow i, \nu \rightarrow j} := 1_{N_i \cup N_\mu \setminus \{\mu\}}(i)1_{N_\nu \cup N_\mu \setminus \{\nu\}}(j)N(\psi^\mu, i, j),
\]

(4)

the update equation of messages given in Eq. (2) becomes a contraction mapping, and therefore, the sum-product algorithm converges to a unique fixed point irrespective of the initial messages.

It is known [6, Proof of Corollary 5] that if the factor graph of \( p(x) \) given in Eq. (1) is a tree, \( \rho(A) = 0 \) holds for the matrix \( A \) defined by Eq. (4).

**Proposition 2:** [6, Proof of Corollary 5] If the factor graph is a tree, Mooij’s sufficient condition for convergence of the sum-product algorithm is activated, i.e., \( \rho(A) < 1 \) holds.

Therefore Proposition 2 and Theorem 1 yield the following well-known fact on the convergence of the sum-product algorithm.

**Corollary 1:** [6, Corollary 5] If the factor graph is a tree, the sum-product algorithm converges to a unique fixed point irrespective of the initial messages.

Theorem 1 can be extended to the case in which each factor \( \psi^\mu \) of \( p(x) \) is nonnegative. In this case, assume in addition that all factors with a single variable are strictly positive\(^2\). Moreover, for each multi-variable factor \( \psi^\mu \) and for all \( i \in \mathcal{I}_\mu \) and \( x_i \in \mathcal{X}_i \), assume that there exists \( x_{\mu,i} \in \prod_{i \in \mathcal{N}_\mu \setminus \{i\}} \mathcal{X}_i \) such that \( \psi^\mu(x_i, x_{\mu,i}) > 0 \). Then by replacing Eq. (3) with

\[
N(\psi^\mu, i, j) := \sup_{h^\beta > 0} \sup_{\alpha, \gamma \in \mathcal{X}_i} \sum_{x_{\mu,i,j}} \frac{\sum_{\beta, \gamma} \psi_{\alpha\beta\gamma}^\mu h_{\beta\gamma}^\mu}{\sum_{\beta, \gamma} \sum_{\alpha} \psi_{\alpha\beta\gamma}^\mu h_{\beta\gamma}^\mu}
\]

(5)

Theorem 1 still holds [6, Theorem 5].

3. Spectral Radius of \( A \) for \( N(\psi^\mu, i, j) = 1 \)

In this section, we assume that \( N(\psi^\mu, i, j) = 1 \) in Eq. (4) and investigate \( \rho(A) \) for a matrix \( A \) defined by Eq. (4). We note in advance that \( N(\psi^\mu, i, j) = 1 \) always holds for the sum-product decoding, which will be shown in Sect. 4.

3.1 Preliminary Observation

Let \( P \) be a permutation matrix. Then \( P^{-1} = P^T \) and the

\(^1\)As Mooij et al. did [6], we also consider the sum-product algorithm with a parallel update scheme, which is realized by updating all messages in parallel.

\(^2\)This assumption is not essential because the single-variable factors that contain zeros can be absorbed into multi-variable factors with the same variable.
transformation $A \rightarrow PAP^T$ permutes the rows and columns of $A$ in the same way. According to Eq. (4), this permutation corresponds to a permutation on an order of edge labels of the factor graph $G$ of $p(x)$ given in Eq. (1). Thus by noting that eigenvalues of $A$ and $PAP^T$ are identical, we conclude that $\rho (A)$ does not depend on an order of the edge labels of $G$.

Moreover, under the assumption that $N(\psi^v, i, j) = 1$, it is obvious from Eq. (4) that $A_{\psi^v,i\rightarrow v\rightarrow j} = 1$ if and only if $1_{N(\psi^v(i))\cap N(\psi^v(j))} \neq 0$. For a complete understanding of the structure of the matrix $A$, it is helpful to notice that the above condition is equivalent to the following condition:

**Lemma 1:** Assume that $N(\psi^v, i, j) = 1$ holds in Eq. (4). If the factor graph $G$ of $p(x)$ given in Eq. (1) consists of a single cycle, then $1$ is an eigenvalue of $A$.

(proof) By regarding the numbers assigned to the edges of $G$ and the numbering of edges of $G$.

Let $v = (v_1, v_2, \ldots, v_{2\ell})$ be a vector of length $2\ell$ defined as $v_1 = v_2 = \cdots = v_\ell = 1$ and $v_{\ell+1} = v_{\ell+2} = \cdots = v_{2\ell} = -1$. Then $Av = v$ holds and therefore we conclude that $1$ is an eigenvalue of $A$.

**3.2 General Case**

Next, we will extend the observation given in Sect. 3.1 to the general case.

**Theorem 2:** Assume that $N(\psi^v, i, j) = 1$ holds in Eq. (4). If the factor graph $G$ of $p(x)$ given in Eq. (1) contains cycles, $\rho (A) \geq 1$.

(proof) Assume that $G$ consists of $e_G$ edges and let $L$ be a minimal cycle of $G$ whose length is $2\ell$. Then by numbering the other edges on $L$ from $1$ to $2\ell$ by the two steps introduced in Subsection 3.1, and by arbitrarily numbering the other edges on $L$ from $2\ell+1$ to $e_G$, $A$ can be expressed as

$$
A = \begin{bmatrix}
M_\ell & O \\
O & M_\ell
\end{bmatrix}
\begin{bmatrix}
U_1 & U_2 \\
U_2 & U_4
\end{bmatrix}
$$

where $O$ is an all-zero matrix of size $\ell \times \ell$, $M_\ell$ is the matrix given in Eq. (6), $U_1$ and $U_2$ are $(e_G - 2\ell) \times \ell$ matrices, and $U_3$ and $U_4$ are $2\ell \times (e_G - 2\ell)$ and $(e_G - 2\ell) \times (e_G - 2\ell)$ matrices, respectively. Then we claim that each row of the sub-matrix $U := [U_1\ U_2]$ of $A$ has no or two $1$’s, and every non-zero row of $U$ has one of $1$’s in $U_1$ and the other in $U_2$.

Since $N(\psi^v, i, j) = 1$ in Eq. (4), as we mentioned in the argument (*) in Sect. 3.1 that $A_{\psi^v,i\rightarrow v\rightarrow j} = 1$ if and only if three edges $(v, j), (\mu, j), (\mu, i)$ in $G$ are serially connecting in this order. Hence an entry of $U$ is $1$ if and only if $(v, j)$ belongs to $L$ and $(\mu, i)$ corresponding to a row of $U$ is connected to $L$ at the factor node $\mu$. This situation is depicted in Fig. 2.

From Fig. 2, it can be verified that the non-zero row of $U$ corresponding to $\mu$ has two $1$’s: one is placed in the column corresponding to $(v, j)$ and the other is placed in the column corresponding to $(v', j')$, respectively. Assume that $(v, j)$ is numbered by one of $[1, 2, \ldots, \ell]$, then $(v', j')$ must be numbered by one of $[\ell + 1, \ell + 2, \ldots, 2\ell]$. Hence one of two $1$’s is placed in $U_1$ and the other is done in $U_2$.

Now, define $v = (v_1, v_2, \ldots, v_{2\ell})$ as

$$
v_i := \begin{cases}
1, & 1 \leq i \leq \ell, \\
-1, & \ell + 1 \leq i \leq 2\ell, \\
0, & 2\ell + 1 \leq i \leq e_G.
\end{cases}
$$

* A cycle is a connected graph where every vertex has exactly two neighbors. The phrase “a cycle in a graph $G$” refers to a subgraph of $G$ that is a cycle.

†† A cycle $L$ in a graph $G$ is said to be minimal if there exists no cycle of $G$ whose vertex set is a proper subset of the vertex set of $L$. 

**Fig. 1** Numbering of $2\ell$ ($\ell = 4, 5$) edges on the cycle. $\square$ and $\circ$ denote factor and variable nodes, respectively.
Then it is clear that $Aw = v$, which implies that 1 is an eigenvalue of $A$, i.e., $\rho(A) \geq 1$.

Under the assumption that $N(\psi^\mu, i, j) = 1$ in Eq. (4), the contraposition of Theorem 2 states that if $\rho(A) < 1$, the factor graph of $p(x)$ given in Eq. (1) does not contain any cycle at all. Hence in the case where $N(\psi^\mu, i, j) = 1$, if Mooij's sufficient condition for convergence of the sum-product algorithm is activated (i.e., $\rho(A) < 1$), the factor graph is a tree.

Finally, by combining the above observation and Proposition 2, we arrive at the following corollary.

**Corollary 2:** Assume that $N(\psi^\mu, i, j) = 1$ holds in Eq. (4). Then Mooij's sufficient condition for convergence of the sum-product algorithm is activated if and only if the factor graph of $p(x)$ given in Eq. (1) is a tree.

---

### 4. Application of Theorem 2

#### 4.1 Sum-Product Decoding

Denote the finite field of two elements by $\mathbb{F}_2 = \{0, 1\}$ and consider a binary linear code $C \subset \mathbb{F}_2^n$ of length $n$. Let $H = [h_{ij}]$ be an $m \times n$ parity check matrix for $C$. For $H$, we define:

$$
\begin{cases}
A_\mu := \{i \mid h_{\mu i} = 1\}, & \mu = 1, 2, \ldots, m, \\
B_i := \{\mu \mid h_{i \mu} = 1\}, & i = 1, 2, \ldots, n.
\end{cases}
$$

In this letter, we assume that $|A_\mu| \geq 2$ holds for all $\mu = 1, 2, \ldots, m$.

Let $x = (x_1, x_2, \ldots, x_n) \in C$ be a transmitted codeword and $y = (y_1, y_2, \ldots, y_n) \in \mathcal{Y}$ be a received sequence, where $\mathcal{Y}$ represents an alphabet of received symbols. In this letter, we assume that the channel is memoryless and its transition probability is denoted by $p(y_i|x_i)$. We also assume that the transmission of a codeword $x$ of $C$ is equiprobable, that is, the a priori probability $q(x)$ of $x$ is expressed as

$$
q(x) = \frac{1}{|C|} \prod_{\mu=1}^m \phi^\mu(x_\mu),
$$

where $x_\mu := (x_i)_{i \in A_\mu}$ and

$$
\phi^\mu(x_\mu) := \begin{cases}
1, & \text{if } \sum_{i \in A_\mu} x_i = 0, \\
0, & \text{otherwise}.
\end{cases}
$$

Finally, by employing Bayes’ rule and Eq. (7), the a posteriori probability $p(x|y)$ is expressed as

$$
p(x|y) = \frac{q(x)p(x|y)}{p(y)} = \kappa \prod_{\mu=1}^m \phi^\mu(x_\mu) \prod_{i=1}^n \phi^i(x_i)
$$

where $\kappa$ is a normalization constant.

The sum-product decoding intends to calculate the approximate marginals $b_i(x_i)$ ($i = 1, 2, \ldots, n$ and $x_i \in \mathbb{F}_2$) for the marginal distribution $p(x|y)$ by the sum-product algorithm. The update equation for messages given in Eq. (2) is translated into

$$
\bar{m}_{\mu\rightarrow i}(x_i) = \kappa_{\mu\rightarrow i} \sum_{\{x_j\in A_{\mu}\mid i\}} \phi^\mu(x_\mu) \prod_{j\in A_{\mu}\setminus\{i\}} p(y_j|x_j) \prod_{v\in B_i\setminus\{i\}} m_{v\rightarrow j}(x_j)
$$

and the approximate marginals are obtained by

$$
b_i(x_i) = \kappa_i p(y_i|x_i) m_{\mu\rightarrow i}(x_i).
$$

Then the transmitted bit $\hat{x}_i$ is estimated by

$$
\hat{x}_i := \begin{cases}
1, & \text{if } b_i(x_i) > b_i(x_i = 0), \\
0, & \text{otherwise}.
\end{cases}
$$

#### 4.2 Value of $N(\psi^\mu, i, j)$ for Sum-Product Decoding

It can be easily verified that for all $i \in A_\mu$ and $x_i \in \mathbb{F}_2$, there exists $x_{\psi^\mu,i}$ such that $\phi^\mu(x_i, x_{\psi^\mu,i}) = 1 > 0$. Hence by restricting the domain of $x_i$ to $\{x_i \in \mathbb{F}_2 \mid \phi^\mu(x_i) = p(y_i|x_i) > 0\}$, Theorem 2 with Eq. (5)\footnote{It can be easily verified that Proposition 2 still holds if $N(\psi^\mu, i, j) = 1$.} instead of Eq. (3) can provide a sufficient condition for convergence of the sum-product decoding.

**Lemma 2:** For the a posteriori probability given by Eq. (9), the following holds.

1. $N(\psi^\mu, i, j) = 1$.
2. $N(\psi^\mu, \psi^\nu, i, j) = N(\psi^\mu, i, j)$.

(proof) (1) For the case of $|A_\mu| \geq 3$: Let

$$
\Delta(\alpha, \alpha', \beta, \beta') := \left| \sum_{\gamma} \phi^\mu_{\alpha\beta\gamma} \phi^\nu_{\beta'\gamma} - \sum_{\gamma} \phi^\nu_{\alpha\beta'\gamma} \phi^\mu_{\beta\gamma} \right|
$$

Since

$$
\begin{align*}
\Delta(0, 0, \beta, \beta') &= \Delta(1, 1, \beta, \beta') = 0, \\
\Delta(0, 1, \beta, \beta') &= \Delta(1, 0, \beta, \beta') = 0.
\end{align*}
$$

\footnote{The expression of $N(\psi^\mu, i, j)$ defined by Eq. (5) can be simplified \cite{6, Eq. (52)} if $\phi^\mu_{\alpha\beta\gamma} \phi^\nu_{\beta'\gamma} + \phi^\nu_{\alpha\beta'\gamma} \phi^\mu_{\beta\gamma} \neq 0$ holds for all the combinations of $\alpha, \alpha', \beta, \beta', \gamma$ satisfying $\alpha \neq \alpha'$ and $\beta \neq \beta'$. However, this is not the case for $p(x|y)$ given in Eq. (9).

---
we have
\[ \sup_{\alpha, \beta, \gamma} \sum_{\beta} \Delta(\alpha, \alpha', \beta, h^\mu) = \sum_{\beta} \Delta(0, 1, \beta, h^\mu). \]

Moreover, it can be verified by the simple calculation that
\[ \sum_{\beta} \sum_{\gamma} \phi_{\alpha \beta \gamma}^\mu h_{\beta \gamma}^\mu = \begin{cases} \frac{\sum_{\gamma} h_{\beta \gamma}^\mu}{\sum_{\gamma} h_{\beta \gamma}^\mu}, & \text{if } \alpha = 0, \\ \sum_{\gamma} h_{\beta \gamma}^\mu, & \text{if } \alpha = 1, \end{cases} \]

where \( \sum_{\gamma}; \text{odd} \) and \( \sum_{\gamma}; \text{even} \) means the summation of \( h_{\beta \gamma} \) for \( \gamma \in \mathbb{Z} \) whose weight is odd and even, respectively. Then we have
\[ \sum_{\beta} \Delta(0, 1, \beta, h^\mu) = \begin{cases} \sum_{\gamma} h_{\beta \gamma}^\mu, & \text{if } \alpha = 0, \\ \frac{\sum_{\gamma} h_{\beta \gamma}^\mu}{\sum_{\gamma} h_{\beta \gamma}^\mu}, & \text{if } \alpha = 1, \end{cases} \]

and we have
\[ \sup_{\alpha, \beta, \gamma} \Delta(\alpha, \alpha', \beta, h^\mu) = \sum_{\beta} \Delta(0, 1, \beta, h^\mu). \]

Hence we also have
\[ N(\phi^\mu, i, j) = \sup_{h^\mu} \frac{1}{2} \sum_{\beta} \Delta(0, 1, \beta, h^\mu) = 1 \]

for \( \mu \) with \( |A_\mu| = 2 \).

By combining above two cases, we conclude that
\[ N(\phi^\mu, i, j) = 1. \]

(2) By noting that
\[ \frac{\sum_{\gamma; \text{odd}} \phi_{\alpha \beta \gamma}^\mu h_{\beta \gamma}^\mu - \sum_{\gamma; \text{even}} \phi_{\alpha \beta \gamma}^\mu h_{\beta \gamma}^\mu}{\sum_{\beta} \sum_{\gamma; \text{odd}} \phi_{\alpha \beta \gamma}^\mu h_{\beta \gamma}^\mu - \sum_{\beta} \sum_{\gamma; \text{even}} \phi_{\alpha \beta \gamma}^\mu h_{\beta \gamma}^\mu} = \Delta(\alpha, \alpha', \beta, h), \]

for \( \mu \) with \( |A_\mu| \geq 3 \), we have
\[ N(\phi^\mu, i, j) = N(\phi^\mu, i, j) = N(\phi^\mu, i, j). \]

Similarly, we can show that
\[ N(\phi^\mu, i, j) = 1 \]

for \( \mu \) with \( |A_\mu| = 2 \).

Lemma 2–(2) means that it is sufficient to consider multi-variable factors in \( p(x|y) \) for the construction of the matrix \( A \). Then we can conclude from Lemma 2–(1) and Corollary 2 that Mooij’s sufficient condition for convergence of the sum-product decoding is activated if and only if the factor graph of the a posteriori probability given in Eq. (9) is a tree.

5. Conclusion

Under the assumption that \( N(\phi^\mu, i, j) = 1 \), we have shown in this letter that Mooij’s sufficient condition for convergence the sum-product algorithm is activated if and only if the factor graph is a tree. By applying this result to the sum-product decoding, we have also shown that Mooij’s sufficient condition for convergence is activated if and only if the factor graph of the a posteriori probability is a tree.

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